CWI Quarterly Volume 12 (3&4) 1999, pp. 369 – 376

On Asymptotic Properties of Infinite Dimensional Stochastic Systems

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Several methods for studying the asymptotic behaviour of infinite dimensional stochastic systems are described and applied to spin models and to stochastic heat and Burgers equations. The results that are presented were developed in [2,3,4,10]. Some extensions are announced as well.

1. Invariant measures for dynamical systems

Let (E, ρ) be a separable, complete metric space, $\mathcal{B}(E)$ the σ-field of its Borel subsets. The space of all probability measures on $\mathcal{B}(E)$, equipped with the metric topology of weak convergence will be denoted by $\mathcal{P}_1(E)$ or shortly $\mathcal{P}_1$. The space $\mathcal{P}_1$ is separable and complete, see [B].

A *transition function* $P_t(x, \Gamma)$, $t \geq 0$, $x \in E$, $\Gamma \in \mathcal{B}(E)$, is a mapping from $[0, +\infty) \times E \times \mathcal{B}(E)$ into $[0, 1]$ such that:

i) $P_t(x, \cdot) \in \mathcal{P}_1$, for all $t \geq 0$ and $x \in E$,

ii) $P_t(\cdot, \Gamma)$ is $\mathcal{B}(E)$-measurable for all $t \geq 0$ and $\Gamma \in \mathcal{B}(E)$,

iii) For all s, $t \geq 0$, $x \in E$ and $\Gamma \in \mathcal{B}(E)$,

$$P_{s+t}(x, \Gamma) = \int_E P_s(x, dy) P_t(y, \Gamma). \tag{1.1}$$

If a transition function P, regarded as a mapping from $[0, +\infty) \times E$ into $\mathcal{P}_1$, is continuous, than P is called *continuous*. In the present paper we will deal only with continuous transition functions.

Assume, for instance, that for each $t \geq 0$ and $x \in E$, the measure $P_t(x, \cdot)$ is concentrated at a point $S_t(x)$. Then (1.1) implies that

$$S_{t+s}(x) = S_s\big(S_t(x)\big),\ t, s \geq 0,\ x \in E, \tag{1.2}$$

and therefore P can be identified with a semigroup of transformations S_t, $t \geq 0$ acting on E. Thus P is a deterministic dynamical system in this case, see [H]. Continuity of P means that the mapping $S_t(x)$, $t \geq 0$, $x \in E$, is continuous. Conversely a deterministic dynamical system satisfying (1.2) determines a transition function P by the formula

$$P_t(x, \cdot) = \delta_{\{S_t(x)\}}(\cdot),\ t \geq 0,\ x \in E. \tag{1.3}$$

If $\nu \in \mathcal{P}_1$ then $P_t^* \nu$, $t \geq 0$ denotes a probability measure given by

$$P_t^* \eta(\Gamma) = \int P_t(x, \Gamma)\eta(dz),\ \gamma \in \mathcal{B}(E). \tag{1.4}$$

Note that the family of transformations P_t^*, $t \geq 0$, from $\mathcal{P}_1$ into $\mathcal{P}_1$ satisfies (1.2) and therefore it is a dynamical system on $\mathcal{P}_1$. If for some $\eta \in \mathcal{P}_1$ and all $t \geq 0$, $P_t^* \eta = \eta$, then η is said to be invariant for P. In particular if P is determined by a dynamical system (S_t) and $z \in E$ is its equilibrium point:

$$S_t(z) = z,\ t \geq 0, \tag{1.5}$$

then $\eta = \delta_{\{z\}}$ is invariant for P. However, there might be many *different* invariant measures for (S_t).

Important examples of deterministic dynamical systems, see [H], are defined through differential equations on Hilbert or Banach spaces or on subsets of such spaces. If, for instance, a linear operator A generates a C_0-semigroup T_t, $t \geq 0$ on a Banach space E and $F : E \to E$ is a continuous mapping and for arbitrary $x \in E$ the *semilinear* equation:

$$\frac{dy}{dt} = Ay + F(y),\ y(0) = x \in F, \tag{1.6}$$

has a unique generalized solution $y^x(t)$, $t \geq 0$ then the mappings $S_t : x \to y^x(t)$ define a continuous transition function on E.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $W(t)$, $t \geq 0$, a Wiener process on a Hilbert space U, with covariance operator Q. The operator Q is assumed to be nonnegative definite on U and bounded. In this paper we are concerned with stochastic evolution equations,

$$dX = \big(AX + F(X)\big)dt + B(X)dW(t),\ X(0) = x = E \tag{1.7}$$

on a Hilbert space E. Under various sets of conditions equation (1.7) has a unique solution $X^x(t), t \geq 0$, and

$$P(t, x, \Gamma) = \mathbb{P}(\omega, X^x(t, \omega) \in \Gamma), t \geq 0, x \in E, \Gamma \in \mathcal{B}(E), \tag{1.8}$$

is a transition function. If $B(x) = 0$ for all $x \in E$, then equation (1.7) reduces to (1.6). Invariant measures for such P are called invariant measures or invariant distributions for (1.7)

It is important to note that if P is a continuous transition function on a compact space E then it has at least one invariant measure. In fact any weak limit as $t \to +\infty$, of measures

$$\nu_t = \frac{1}{t}\int_0^t P_s^* \nu ds,\ t \geq 0, \tag{1.9}$$

where ν is any element of $\mathcal{P}_1(E)$, is invariant for P. Compactness of E implies tightness of the family $\{\nu_t,\ t > 0\}$ and therefore the existence of the weak limit of $\{\nu_{t_n}\}$ for some $t_n \uparrow +\infty$.

If the space E is only locally compact then the existence of an invariant measure for an equation follows from the boundedness in probability of its solutions. In particular if $E = \mathbb{R}^d$, $A = 0, B = 0$, F is a bounded Lipschitz mapping and for some $x \in F$, the set $\{S_t(x),\ t \geq 0\}$ is bounded then there exists an invariant measure for the corresponding transition function P. If the Banach space is infinite dimensional the boundedness of all solutions of an equation does not imply the existence of an invariant measures. The following result due to I. Vrkoč [13] is instructive.

Theorem 1.1 (Vrkoč) *There exists a bounded, Lipschitz mapping F on a Hilbert space E, $\dim E = +\infty$, such that $F(z) \neq 0$ for all $z \in E$ and all solutions $y^x(t)$, $t \geq 0$, of*

$$\frac{dy}{dt} = F(y),\ y(0) = x, \tag{1.10}$$

are bounded and tend weakly to 0 as $t \to +\infty$. In particular there are no invariant measures for (1.10).

Therefore in the case when the state space is an infinite dimensional Hilbert space new principles are needed . Some of them are discussed in the following sections.

2. Dissipativity and spin systems

One way of establishing the existence of invariant measures when E is not locally compact is to show that the system under considerations has a *dissipation* property. We describe here a class of equations for which this is the case. They are of the form:

$$dX = \big(AX + F(X)\big)dt + BdW(t),\ X(0) = x = E, \tag{2.1}$$

where E is a separable Hilbert space, W a Wiener process on a Hilbert space U and B is a bounded linear operator from U into H. The mappings A and F will satisfy appropriate dissipativity conditions which we formulate now.

Let $(E, ||\cdot||_E)$ be a Banach space and E^* its dual. For arbitrary $x \in E$, the subdifferential $\partial||x||_E$ of the norm $||\cdot||_E$ at x is given by the formula:

$$\partial||x||_E = \{x^* \in E^* : ||x+y||_E - ||x||_E \geq x^*(y),\ \forall_y \in E\}. \tag{2.2}$$

A mapping $G : D(G) : E \to E$ is dissipative if for arbitrary $x, y \in D(G)$ there exists $z \in \partial||x-y||_E$ such that,

$$z^*\big(G(x) - G(y)\big) \leq 0. \tag{2.3}$$

If in addition, for some $\alpha > 0$, the mapping $I - \alpha G$ is surjective, G is called m-dissipative. If $K \subset E$ is a Banach space continuously embedded into E then the part G_K of G in K is the restriction of G to the set

$$D(G_K) = \{x \in D(G) \cap K : G(x) \in K\}. \tag{2.4}$$

We will need the following assumption:

(A.1) E is a Hilbert space and a reflexive Banach space K is continuously embedded into E.

To obtain existence of a solution to (2.1) we will require that the process,

$$W_A(t) = \int_0^t T(t-s)dW(s), \ t \geq 0, \tag{2.5}$$

is regular in the following sense:

(A.2) The process W_A takes values in $D(F_K)$ and for each $t > 0$,

$$\sup_{0 \leq s \leq t}\Big(||W_A(t)||_K + \big|\big|F_K\big(W_A(t)\big)\big|\big|_K\Big) < +\infty, \ \mathbb{P} \ a.s. \tag{2.6}$$

Note that the process W_A is a generalized solution of the linear version of the equation (2.1).

The following theorem is taken from [3].

THEOREM 2.1. *Assume that the operator A generates a C_0-semigroup $T(t)$, $t \geq 0$ on E and that assumptions* (A.1) *and* (A.2) *hold. Let moreover for some constants η_1, η_2 operators $A + \eta_1 I$, $F + \eta_2 I$ are m-dissipative on E and on K and F maps bounded sets in K into bounded sets in E. Then the equation* (2.1) *has a unique generalized solution. If, in addition, $w = \eta_1 + \eta_2 > 0$ and*

$$\sup_{t \geq 0}\Big(\mathbb{E}||W_A(t)||_E + \big|\big|F\big(W_A(t)\big)\big|\big|_E\Big) < +\infty, \tag{2.7}$$

then there exists a unique invariant measure μ for the transition function P determined by (2.1) *and for all bounded, Lipschitz continuous functions φ on E one has*

$$\begin{aligned} &\left|\int_E \varphi(y)P_t(x,dy) - \int_E \varphi(y)\mu(dy)\right| \\ &\leq (c + 2||x||_E)e^{-wt}||\varphi||_{\text{Lip}}, \ x \in E, \ t \geq 0, \end{aligned} \tag{2.8}$$

where

$$c = \sup_{t \geq 0}\mathbb{E}\Big(||W_A(t)||_E + \frac{1}{w}\big|\big|F\big(W_A(t)\big)\big|\big|_E\Big). \tag{2.9}$$

As an illustration we shall apply this theorem to *spin systems*, see [3].

Let $\mathbb{Z}$ be the set of all integers and $\mathbb{Z}^d$ the lattice which elements can be interpreted as atoms. A *configuration* x is any real function on $\mathbb{Z}^d$. Spin systems on $\mathbb{Z}^d$ are determined by an infinite matrix $(a_{\gamma j})_{\gamma, j\in\mathbb{Z}^d}$ and a real function $f:\mathbb{R}\to\mathbb{R}$ called respectively a *global interaction matrix* and a *local interaction function*. Let $X(t)$ be the configuration of the spin system at time $t\geq 0$. The process $X(t)=\big(X_\gamma(t)\big)_{\gamma\in\mathbb{Z}^d}$, $t\geq 0$, satisfies an infinite system of equations:

$$\begin{aligned} &(2.10)\quad dX_\gamma(t)=\Big(\sum_j a_{\gamma j}X_j(t)+f\big(X_\gamma(t)\big)\Big)dt+dW_\gamma(t)\\ &\qquad\qquad X_\gamma(0)=x_\gamma,\ \gamma\in\mathbb{Z}^d,\ t\geq 0,\end{aligned}$$

in which $W_\gamma, \gamma\in\mathbb{Z}^d$ are real Brownian motions. Invariant measures for (2.10) are called *Gibbs measures.*

The preceding theorem will be applied with the operators A and F defined as follows:

$$\begin{aligned} A(x_\gamma)&=\left(\sum_{j\in\mathbb{Z}^d}a_{\gamma j}x_j\right),\ x=(x_\gamma)\in E,\\ F(x_\gamma)&=\big(f(x_\gamma)\big),\ x=(x_\gamma),\end{aligned}$$

where $E=l^2_\rho(\mathbb{Z}^d)$ is a Hilbert space of sequences (x_γ), summable with respect to a positive weight function $\rho:\mathbb{Z}^d\to\mathbb{R}^+$. Let $U=l^2(\mathbb{Z}^d)$ and B be the inclusion of U into E. We will restrict to the weight

$$\rho_\kappa(\gamma)=\frac{1}{1+\kappa|\gamma|^r},\ \gamma\in\mathbb{Z}^d$$

where $\kappa>0$ and $r>d$.

Theorem 2.2. *Assume that $f=f_0+f_1$ where f_1 is Lipschitz and f_0 is continuous, decreasing such that for some $s\geq 1$ and $c_0>0$:*

$$|f_0(\xi)|\leq c_0(1+|\xi|^s),\ \xi\in\mathbb{R}^1.$$

Let in addition, for some $\eta>0$, $w>0$, the operator $A+\eta I$, restricted to $l^2(\mathbb{Z}^d)$ be dissipative and $\eta-||f_1||_{\mathrm{Lip}}>w$. Then there exists a unique invariant measure μ for (2.10) *in any $E=l^2_{\rho_\kappa}(\mathbb{Z}^d)$, $\kappa>0$. Moreover, there exists $c>0$ such that*

$$\left|\int_E\varphi(y)P_t(x,dy)-\int_E\varphi(y)\mu(dy)\right|\leq(c+2||x||_E)e^{-wt}||\varphi||_{\mathrm{Lip}}.$$

Theorem 2.2 has natural extensions to spin systems on $\mathbb{R}^d$. In particular we have the following theorem from [3] concerned with an $\mathbb{R}^d$-analog of (2.10):

$$\begin{aligned} &(2.11)\quad dX(t,\xi)=\big[(\Delta-\eta)X(t,\xi)+f\big(X(t,\xi)\big)\big]dt+dW(t,\xi)\\ &\qquad\qquad X(0,\xi)=x(\xi),\ \xi\in\mathbb{R}^d,\ t>0.\end{aligned}$$

In the formulation of the theorem $L^2_{\rho_r}(\mathbb{R}^d)$ denotes the space of all Borel functions x on $\mathbb{R}^d$ such that:

$$||x||_r^2 = \int_{\mathbb{R}^d} |x(\xi)|^2 \frac{1}{1+|\xi|^r} d\xi < +\infty,$$

where $r > d$. We will assume also that the Wiener process W is space homogeneous:

$$\mathbb{E}W(t,\xi)W(s,\eta) = t \wedge s q(\xi - \eta), \; t \geq 0, \; \xi, \eta \in \mathbb{R}^d, \tag{2.12}$$

where q is a positive definite, continuous function.

THEOREM 2.3. *Assume that q is a continuous and integrable positive definite function and that the function f satisfies the same conditions as in Theorem* 2.2. *Then there exists a unique invariant measure μ for* (2.11) *in $E = L^2_{\rho_r}(\mathbb{R}^d)$, for any $r > d$. Moreover, there exists $c > 0$ such that*

$$\left| \int_E \varphi(y) P_t(x, dy) - \int_E \varphi(y)\mu(dy) \right| \leq (c + 2||x||_E) e^{-wt} ||\varphi||_{\mathrm{Lip}}.$$

3. THE COMPACTNESS METHOD AND BURGERS EQUATION

Theorem 2.1 cannot be directly extended to systems with state dependent diffusion operators,

$$\begin{aligned} dx &= \big(AX + F(X)\big)dt + B(X)dW(t) \\ X(0) &= x. \end{aligned} \tag{3.1}$$

The dissipativity method imposes additional conditions on the noise process and on the coefficients, see [2] and [4]. More convenient in the present situation is the compactness method from [2] which was adapted in [10] to stochastic heat equations of the form:

$$\begin{aligned} dX(t,\xi) &= \Delta X(t,\xi)dt + b\big(X(t,\xi)\big)dW(t,\xi) \\ X(0,\xi) &= x(\xi) \end{aligned} \tag{3.2}$$

in the space $E = L^2_{\rho_\kappa}(\mathbb{R}^d)$ introduced in Section 2. Let T_t, $t \geq 0$, be the semigroup on $L^2_{\rho_\kappa}(\mathbb{R}^d)$ generated by Δ. Thus

$$T_t x(\xi) = \int_{\mathbb{R}^d} p_t(\xi - \eta) x(\eta) d\eta,$$

where

$$p_t(\xi) = \frac{1}{\sqrt{(4\pi t)^d}} e^{-\frac{|\xi|^2}{4t}}, \; \xi \in \mathbb{R}^d, \; t > 0.$$

The following lemmas are important ingredients of the compactness method, see [10].

Lemma 3.1. *If $r, \hat{r} > d$ and $r > \hat{r}+d$ then for all $t > 0$ the operators T_t, $t \geq 0$, are compact from $L^2_{\rho_{\hat{r}}}(\mathbb{R}^d)$ into $L^2_{\rho_r}(\mathbb{R}^d)$.*

Lemma 3.2. *If the assumptions of Lemma 3.1 are satisfied and there exists a $\hat{x} \in L^2_{\rho_{\hat{r}}}(\mathbb{R}^d)$ such that the solution $X^{\hat{x}}(t)$, $t \geq 0$, of (3.2) is bounded in probability in $L^2_{\rho_{\hat{r}}}(\mathbb{R}^d)$, then there exists an invariant measure for (3.2) in the space $L^2_{\rho_r}(\mathbb{R}^d)$.*

From these lemmas one can deduce the following important result. Let the covariance function q be determined by (2.12) and let Γ denote the Euler gamma function.

Theorem 3.1. *Assume that the function b is Lipschitz continuous with Lipschitz constant $c > 0$, and that $d \geq 3$, $r > 2d$ and q is a positive definite function such that*

$$\int_{\mathbb{R}^d} q(\xi)|\xi|^{2-d} d\xi \;<\; \frac{1}{c^2} \cdot \frac{4\pi^{d/2}}{\Gamma\left(\frac{d}{2}-1\right)}. \tag{3.3}$$

Then there exists an invariant measure for (3.2) in $L^2_{\rho_r}(\mathbb{R}^d)$.

Theorem 3.1 can be applied to the so called *stochastic Burgers equation*, see [7] and [10]. This is a vector equation of the form:

$$du_k(t,\xi) = \left(\Delta u_k(t,\xi) - \sum_{l=1}^{d} \frac{\partial u_k(t,\xi)}{\partial \xi_l} u_l(t,\xi)\right) dt + d\frac{\partial}{\partial_{\xi_k}} W(t,\xi), \tag{3.4}$$

where $k = 1, 2, \ldots, d$, $\xi \in \mathbb{R}^d$. If X denotes a solution to (3.2) with a linear function $b(\sigma) = c\sigma$, $\sigma \in \mathbb{R}^1$ then the process $u(t,\xi) = \big(u_1(t,\xi), \ldots, u_d(t,\xi)\big)$ defined by

$$u_k(t,\xi) = -\frac{\partial}{\partial \xi_k}[\ln X(t,\xi)],\; k = 1, 2, \ldots, d,\; t > 0, \xi \in \mathbb{R}^d, \tag{3.5}$$

is, at least formally, a solution to (3.4), see [7]. The transformation (3.5) connecting the process X and u is called the *Cole-Hopf transform*.

As a corollary one can show that under the assumptions of Theorem 3.1, there exists an invariant measure for the stochastic Burgers equation. For more details we refer to [10]. A longer, computational proof of a similar result can be found in [7]. The more abstract proof indicated here is considerably shorter. The special case of one and two dimensional systems is the subject of [12]. It is a challenging question to derive theorems from [12] using the compactness method, see also the comments in [11].

4. Notes

Extensions to more general spin systems can be found in [3,4,11]. In particular, in [11] the existence of invariant measures for general stochastic heat equations

$$\begin{aligned} &dX(t,\xi) = (\Delta X(t,\xi) + f(X(t,\xi)))dt + b\big(X(t,\xi)\big)dW(t,\xi) \\ &X(0,\xi) = x(\xi)\,, \end{aligned} \tag{4.1}$$

as well as for stochastic wave equations

$$dX(t,\xi) = Y(t,\xi)dt, \tag{4.2}$$

$$dY(t,\xi) = (\Delta X(t,\xi) + CY(t,\xi)))dt + b\big(X(t,\xi)\big)dW(t,\xi) \tag{4.3}$$

$$X(0,\xi) = x(\xi)$$

are investigated. For both equations the noise process is space homogeneous with a general covariance function. Recent existence and regularity results, developed in [6,8,9] are used in an essential way.

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